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Technical Note

On the perturbation method for the Stefan problem with time-dependent boundary conditions

J. Caldwell *, Y.Y. Kwan

Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong

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Abstract

Perturbation methods are developed for Stefan problems with time-dependent boundary conditions. The methods are applied to melting of ice in the half-plane, outward spherical solidification and outward cylindrical solidification of a saturated liquid. The results are shown to compare well with those obtained by other numerical methods.

Keywords: Moving boundary; Stefan problem; Perturbation method

1. Introduction

Like other branches of engineering, heat transfer analysis during the past two decades has been developed on numerical simulation. However, approximate analytical methods have continued to develop and provide useful solutions to a variety of problems. One is the method of perturbation expansion. A review of the literature in this specific area has appeared in [1].

Perturbation methods have been successfully applied to Stefan problems with simple boundary conditions in different geometries. The perturbation solutions for the planar solidification of a saturated liquid with convection at the wall has been found by Pedroso and Domoto [2], and Huang and Shih [3]. On the other hand, Pedroso and Domoto [4], and Stephan and Holzknecht [5] have found the perturbation solutions for outward spherical and cylindrical solidifications. The present note develops the perturbation methods for the phase change problems with time-dependent boundary conditions and demonstrates the high accuracy when compared to other numerical solutions.

2. Melting in the half-plane

Consider the melting of ice initially at its freezing temperature T_f in the half-plane $x > 0$ subject to a time-dependent temperature change at $x = 0$. The governing equation for the process is

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < y(t), \quad t > 0 \tag{1}$$

with boundary conditions:

$$T(x = 0, t) = f(t), \quad T(x = y(t), t) = 0, \tag{2}$$

$$\frac{dy}{dt} = -\alpha \left(\frac{\partial T}{\partial x} \right)_{x=y(t)}, \tag{3}$$

where T is the temperature, x is the space variable, $y(t)$ is the position of the moving boundary and $\alpha = c(T_f - T_{ref})/L$ is the Stefan number *Ste*, c is the specific heat content, L is the latent heat and T_{ref} is the reference temperature selected such that $f(t = 0) = 1$. Since $y(t)$ is expected to be a monotonic function of t, we may replace t by y as the second independent variable. By making use of Eq. (3), Eq. (1) can be written as

$$\frac{\partial^2 T}{\partial x^2} = -\alpha \frac{\partial T}{\partial y} \left(\frac{\partial T}{\partial x} \right)_{x=y}. \tag{4}$$

* Corresponding author. Tel.: +852-2788-8646/8934; fax: +852-2788-8561.
E-mail address: majimca@cityu.edu.hk (J. Caldwell).

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Nomenclature

L	latent heat	*Greek symbol*	
T	temperature	α	Stefan number, *Ste*, dimensionless
c	specific heat content	*Subscripts*	
r	radial variable	f	freezing
t	time variables	ref	reference
x	space variable		
$y(t)$	position of moving boundary		

The boundary condition at $x = 0$ is written as

$$T = f(t) = F(y) \text{ on } x = 0. \quad (5)$$

Now, we make the assumption that α is small, and derive a three-term perturbation solution of the form

$$T(x,y) = T_0(x,y) + \alpha T_1(x,y) + \alpha^2 T_2(x,y). \quad (6)$$

Substituting Eq. (6) into Eqs. (2) and (4), the governing equations for T_0, T_1, and T_2 will be

$$\begin{aligned}
\alpha^0 &: \frac{\partial^2 T_0}{\partial x^2} = 0, \\
&\quad T_0(x=0,y) = F(y), \quad T_0(x=y,y) = 0. \\
\alpha &: \frac{\partial^2 T_1}{\partial x^2} = -\frac{\partial T_0}{\partial y}\left(\frac{\partial T_0}{\partial x}\right)_{x=y}, \\
&\quad T_1(x=0,y) = 0, \quad T_1(x=y,y) = 0. \\
\alpha^2 &: \frac{\partial^2 T_2}{\partial x^2} = -\frac{\partial T_0}{\partial y}\left(\frac{\partial T_1}{\partial x}\right)_{x=y} - \frac{\partial T_1}{\partial y}\left(\frac{\partial T_0}{\partial x}\right)_{x=y}, \\
&\quad T_2(x=0,y) = 0, \quad T_2(x=y,y) = 0.
\end{aligned} \quad (7)$$

The solutions are, respectively,

$$\begin{aligned}
T_0(x,y) &= F(y)(1-z), \\
T_1(x,y) &= \tfrac{1}{6}F(y)z(z-1)[F(y)(z+1) - F'(y)y(z-2)], \\
T_2(x,y) &= \tfrac{-1}{360}F(y)z(z-1)\Big[F(y)^2(z+1)(9z^2+19) \\
&\quad + 10F'(y)^2y^2(z+4) \\
&\quad + 5F(y)F'(y)y(3z^2+5z+17) \\
&\quad + F(y)F''(y)y^2(z-2)(3z^2-6z-4)\Big],
\end{aligned} \quad (8)$$

where $z = x/y$. Thus, the position of the moving boundary follows the equation

$$\begin{aligned}
\frac{dy}{dt} &= -\alpha\left(\frac{\partial T_0}{\partial x} + \alpha\frac{\partial T_1}{\partial x} + \alpha^2\frac{\partial T_2}{\partial x}\right)_{x=y} \\
&= \frac{\alpha}{y}F(y) - \alpha^2 F(y)\left[\frac{1}{6}F'(y) + \frac{1}{3y}\right] \\
&\quad + \alpha^3 F(y)\left[\frac{7}{45y}F(y)^2 + \frac{5}{36}F'(y)^2 y \right. \\
&\quad \left. + \frac{25}{72}F(y)F'(y) - \frac{13}{360}F(y)F''(y)y\right].
\end{aligned} \quad (9)$$

Substituting back $f(t)$ for $F(y)$, with the relations

$$\frac{dF(y)}{dy} = \frac{df(t)}{dt}\left(\frac{dy}{dt}\right)^{-1}, \quad \frac{d^2F(y)}{dy^2} = \frac{d^2f(t)}{dt^2}\left(\frac{dy}{dt}\right)^{-2},$$

Eq. (9) can be rewritten in the form

$$\left(\frac{dy}{dt}\right)^3 + a(t,y)\left(\frac{dy}{dt}\right)^2 + b(t,y)\frac{dy}{dt} + c(t,y) = 0, \quad (10)$$

where

$$\begin{aligned}
a(t,y) &= -\frac{\alpha f(t)}{y}\left[1 - \frac{\alpha}{3}f(t) + \frac{7\alpha^2}{45}f(t)^2\right], \\
b(t,y) &= \alpha^2 f(t)f'(t)\left[\frac{1}{6} - \frac{25\alpha}{72}f(t)\right], \\
c(t,y) &= -\alpha^3 f(t)y\left[\tfrac{5}{36}f'(t)^2 - \tfrac{13}{360}f(t)f''(t)\right].
\end{aligned}$$

By solving the cubic equation (10), the value of dy/dt is obtained and y can be found by numerical integration, while the temperature distribution can be obtained by substituting Eq. (8) into Eq. (6).

Since there is a singularity in Eq. (10) when y is close to 1, an approximation is needed. As this occurs only for small t, one possible simplification is to ignore the derivatives of $f(t)$. In this case, Eq. (10) becomes

$$\frac{dy}{dt} = \frac{\alpha}{y}\left[1 - \frac{\alpha}{3}f(t) + \frac{7\alpha^2}{45}f(t)^2\right] \quad (11)$$

and the solution is

$$y(t) = \left\{2\alpha\int_0^t\left[1 - \frac{\alpha}{3}f(\tau) + \frac{7\alpha^2}{45}f(\tau)^2\right]d\tau\right\}^{1/2}. \quad (12)$$

3. Outward spherical solidification

Consider the outward spherical solidification of a saturated liquid due to low temperature at the boundary. The problem can be formulated as

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial^2(rT)}{\partial r^2}, \quad 1 < r < y(t), \quad t > 0, \quad (13)$$

$$T(r=1,t)=f(t), \quad T(r=y(t),t)=1, \tag{14}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\alpha\left(\frac{\partial T}{\partial r}\right)_{r=y(t)}. \tag{15}$$

As before, we use y as the second independent variable instead of t. We rewrite Eqs. (13) and (14) as

$$\frac{1}{r}\frac{\partial^2(rT)}{\partial r^2}=\alpha\frac{\partial T}{\partial y}\left(\frac{\partial T}{\partial r}\right)_{r=y}, \tag{16}$$

$$T(r=1,y)=F(y), \quad T(r=y,y)=1. \tag{17}$$

Assume that there is a three-term perturbation solution of the form

$$T(r,y)=T_0(r,y)+\alpha T_1(r,y)+\alpha^2T_2(r,y), \tag{18}$$

by substituting Eq. (18) into Eqs. (16) and (17), T_0, T_1 and T_2 are found to satisfy:

$$\begin{aligned}
\alpha^0: &\ \frac{1}{r}\frac{\partial^2(rT)}{\partial r^2}=0,\\
&\ T_0(r=1,y)=F(y), \quad T_0(r=y,y)=1.\\
\alpha: &\ \frac{1}{r}\frac{\partial^2(rT)}{\partial r^2}=-\frac{\partial T_0}{\partial y}\left(\frac{\partial T_0}{\partial x}\right)_{r=y},\\
&\ T_1(r=1,y)=0, \quad T_1(r=y,y)=0.\\
\alpha^2: &\ \frac{1}{r}\frac{\partial^2(rT)}{\partial r^2}=-\frac{\partial T_0}{\partial y}\left(\frac{\partial T_1}{\partial x}\right)_{r=y}-\frac{\partial T_1}{\partial y}\left(\frac{\partial T_0}{\partial x}\right)_{r=y},\\
&\ T_2(r=1,y)=0, \quad T_2(r=y,y)=0.
\end{aligned} \tag{19}$$

$y(t)$ is found to satisfy the same form of Eq. (10), with

$$a(t,y)=\frac{-\alpha f(t)[45y^3-15\alpha f(t)y^2+\alpha^2f(t)^2(6y+1)]}{45y^4(y-1)},$$

$$b(t,y)=\frac{\alpha^2f(t)f'(t)[60y+7-\alpha f(t)(10y+1)]}{360y^3},$$

$$c(t,y)=\frac{-\alpha^3f(t)(y-1)[17f'(t)^2+7f(t)f''(t)]}{360y^3}.$$

These can easily be obtained through a symbolic mathematics program.

Again, there is a singularity for small t. Since the expression is much more complicated than in the plane geometry case, we will ignore the change in $f(t)$ in deriving the initial approximation. Suppose $f(t=0)=1$, then $y(t)$ satisfies

$$y(y-1)\left[1-\frac{\alpha}{3y}+\frac{\alpha^2(6y+1)}{45y^3}\right]^{-1}\mathrm{d}y=\mathrm{d}t.$$

Expanding the term in parentheses binomially and retaining terms up to $O(\alpha^2)$, we have

$$\left[y(y-1)+\frac{\alpha}{3}(y-1)-\frac{\alpha^2}{45}\left(1-\frac{1}{y^2}\right)\right]\mathrm{d}y=\mathrm{d}t. \tag{20}$$

Integrating Eq. (20), gives

$$t=(y-1)^2\left(\frac{2y+\alpha+1}{6}-\frac{\alpha^2}{45y}\right). \tag{21}$$

4. Outward cylindrical solidification

In the case of outward cylindrical solidification, the corresponding governing equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\alpha\frac{\partial T}{\partial y}\left(\frac{\partial T}{\partial r}\right)_{r=y} \tag{22}$$

subject to boundary conditions Eqs. (15) and (17). Assuming a three-term perturbation solution of the form Eq. (18), the corresponding system for T_0, T_1 and T_2 will be

$$\begin{aligned}
\alpha^0: &\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=0,\\
&\ T_0(r=1,y)=F(y), \quad T_0(r=y,y)=1.\\
\alpha: &\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=-\frac{\partial T_0}{\partial y}\left(\frac{\partial T_0}{\partial x}\right)_{r=y},\\
&\ T_1(r=1,y)=0, \quad T_1(r=y,y)=0.\\
\alpha^2: &\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=-\frac{\partial T_0}{\partial y}\left(\frac{\partial T_1}{\partial x}\right)_{r=y}-\frac{\partial T_1}{\partial y}\left(\frac{\partial T_0}{\partial x}\right)_{r=y},\\
&\ T_2(r=1,y)=0, \quad T_2(r=y,y)=0.
\end{aligned} \tag{23}$$

$y(t)$ is found to satisfy the same form of Eq. (10), with

$$\begin{aligned}
a(t,y)=\frac{-\alpha f(t)}{128y^5\ln y^7}\{&128y^4\ln y^6-32\alpha f(t)y^2\ln y^3(2y^2\ln y^2\\
&-2y^2\ln y+y^2-1)+\alpha^2f(t)^2[y^4(48\ln y^4-112\ln y^3\\
&+146\ln y^2-111\ln y+40)-16y^2(2\ln y^2\\
&-5\ln y+5)+10\ln y^2+31\ln y+40]\},
\end{aligned}$$

$$\begin{aligned}
b(t,y)=\frac{\alpha^2f(t)}{128y^4\ln y^6}\{&32y^2\ln y^3(y^2\ln y+\ln y-y^2+1)\\
&-\alpha f(t)f'(t)[y^4(66\ln y^3-149\ln y^2+140\ln y\\
&-56)+y^2(40\ln y^3-8\ln y^2-80\ln y+112)\\
&-6\ln y^3-19\ln y^2-60\ln y-56]\},
\end{aligned}$$

$$\begin{aligned}
c(t,y)=\frac{-\alpha^3f(t)}{128y^3\ln y^5}\{&f(t)f''(t)[2\ln y^2(3y^4+4y^2+3)-13\\
&\times\ln y(y^4-1)+8(y^2-1)^2]+f'(t)^2[2\ln y^2(7y^4\\
&+12y^2+7)-29\ln y(y^4-1)+16(y^2-1)^2]\}.
\end{aligned}$$

The initial approximation can be found in the same way as in the case of outward spherical solidification, and the resulting equation is

$$t = \frac{\alpha}{4}(2y^2 \ln y - y^2 + 1) + \frac{\alpha^2(y^2 \ln y + \ln y - y^2 + 1)}{4 \ln y} + \frac{\alpha^3[y^4(8 \ln y^3 - 20 \ln y^2 + 21 \ln y - 8) - 16y^2(\ln y - 1) - 5 \ln y - 8]}{128 y^2 \ln y^4}. \quad (24)$$

5. Numerical results

We present one numerical example for each geometry, here, for $\alpha = 0.2$. Since there are only a few examples with known analytic solutions, we will compare the numerical results with those obtained by other numerical methods. The test example for melting in the half-plane has the boundary condition $T(x = 0, t) = f(t) = (1 - 0.2t)$ which was studied by Mennig and Özişik [6]. Table 1 shows the position of the moving boundary at different times. From the table, the results by the perturbation method agree well with those obtained by Mennig and Özişik and differ by at most 0.3% over the time values listed.

Table 1
Position of moving boundary at different times for melting in the half-plane

t	y from [6]	y from perturbation
0.1068	0.2000	0.1994
0.2441	0.3000	0.2995
0.4425	0.4000	0.3996
0.7094	0.5000	0.4995
1.057	0.6000	0.5994
1.508	0.7000	0.6995
2.103	0.8000	0.7995
2.951	0.9000	0.9000

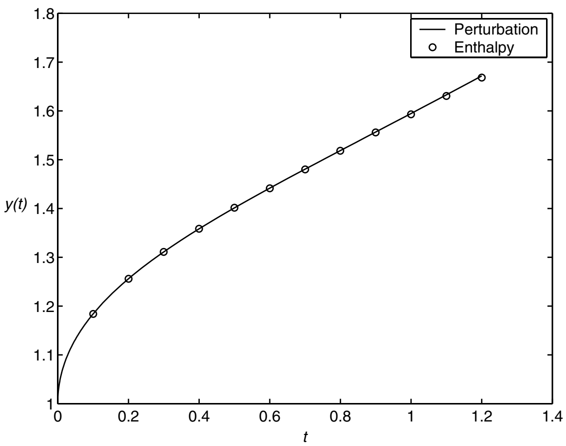

Fig. 1. Position of moving boundary $y(t)$ at different times for outward spherical solidification.

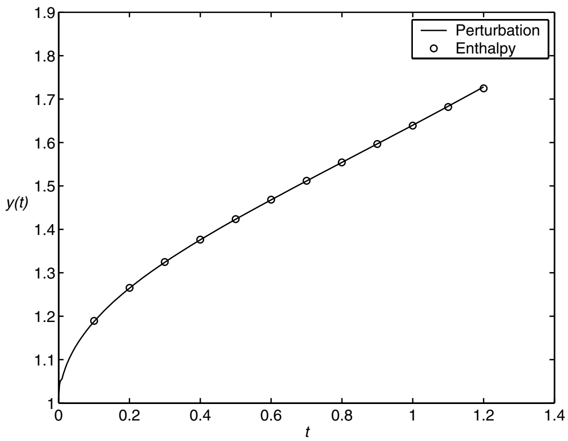

Fig. 2. Position of moving boundary $y(t)$ at different times for outward cylindrical solidification.

In the cases of outward spherical and cylindrical solidifications, the test example with boundary condition $T(r = 1, t) = f(t) = -t^2$ is used. The problem is solved by both the perturbation method and the enthalpy method. Readers may refer to Caldwell and Kwan [7] for details of the enthalpy method. The results obtained by the two methods for the two geometries are shown in Figs. 1 and 2. From the figures, clearly the results from the two methods agree well. In fact, over the range $0 < t < 1.2$, the maximum percentage difference between the results is approximately 0.2%.

6. Conclusion

The perturbation method is used to solve the Stefan problems with time-dependent boundary conditions for different geometries. The results agree well with those from other numerical methods.

Although the results presented are only for $\alpha = 0.2$ here, we can conclude from the application of the method to test cases with time-independent boundary conditions in the literature that the method proposed is expected to perform well for α up to 0.5. If more terms are included in the perturbation series, the method is expected to work well for even larger α. Though the algebra will become more complicated, it is still worth doing in practice since the formulae can be reused for different boundary conditions.

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